Pergamon

Int. J. Solids Structures Vol. 35, No. 20, pp. 2695–2714, 1998

0020-7683/98 $19.00 + .00

PII: S0020-7683(97)00172-8

THE WEDGE SUBJECTED TO TRACTIONS PROPORTIONAL TO r^n: A PARADOX RESOLVED

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(*Received* 29 *January* 1997; *in revised form* 12 *June* 1997)

Abstract—The classical two-dimensional solution for the stress distribution in an elastic wedge subjected to tractions proportional to $r^n(n \geqslant 0)$ becomes infinite when the wedge angle 2α and the constant n satisfy the definite relations, this is a paradox. For $n = 0$ it was resolved by Dempsey [*Journal of Elasticity* **11**, 1–10 (1981)] and Ting [*Journal of Elasticity* **14**, 235–247 (1984)], for $n > 0$ and $2\alpha = \pi$ or 2π it was resolved by Wang [*Acta Mechanica Sinica* **18**(3), 242–252 (1986)]. However, the above investigations provided only a little resolution of it. In this paper all the cases of the paradox have been studied by employing the complex variable method, and the corresponding bounded solutions are obtained. Moreover, the secondary paradox is discovered in the problem, i.e., the initial solution for the paradox is still infinite for some special values of (n, α), and this is also resolved here. From the results obtained it can be observed that the stress distribution contains a $r^n(\ln r)$ term for the paradox and a $r^n(\ln r)^2$ term more for the secondary paradox. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

It is well-known that the two-dimensional solution for the stress distribution in an elastic wedge subjected to a concentrated couple at the vertex becomes infinite when the wedge angle 2α is equal to $2\alpha_*$ where $\tan 2\alpha_* = 2\alpha_*$ (Timoshenko and Goodier, 1970), this paradox was resolved by Sternberg and Koiter (1958) and Ting (1985), after that, Dundurs (1989) and Markenscoff (1994) made further investigations again.

For a wedge subjected to tractions proportional to $r^n(n \geqslant 0)$ on the surfaces, the classical solution obtained through the method put forward by Timoshenko and Goodier (1970) also becomes infinite when 2α and n satisfy the definite relations, i.e., $(n+1)\sin 2\alpha + \sin 2(n+1)\alpha = 0$ (for symmetric deformations) or $(n+1)\sin 2\alpha - \sin 2(n+1)\alpha = 0$ (for antisymmetric deformations). For the special case $n = 0$, Dempsey (1981) obtained the solutions which are bounded at $2\alpha = \pi$ or 2π (for symmetric deformations) and $2\alpha_*$ (for antisymmetric deformations), Ting (1984) provided the solutions which are bounded for 2α near and equal to π, 2π and $2\alpha_*$ through the superposition of the homogeneous solutions. For the special case $n > 0$ and 2α near and equal to π (when $n = 1, 2, 3, \ldots$) or 2π (when $n = 1/2, 1, 3/2, 2, \ldots$), Wang (1986) obtained the bounded solutions by adopting the same method as Ting did. As for the roots of the above two transcendental equations, England (1971) and Moffatt and Duffy (1980) and Ting (1984) have made an exhaustive study of

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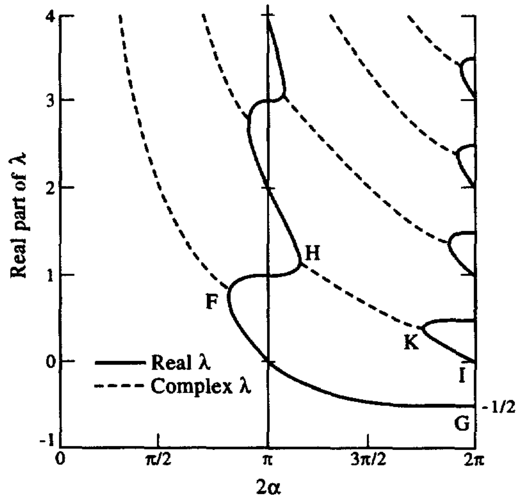

Fig. 1. Roots of $P(\lambda,\alpha) = (\lambda+1)\sin 2\alpha + \sin 2(\lambda+1)\alpha = 0$. Taken from Ting (1984).

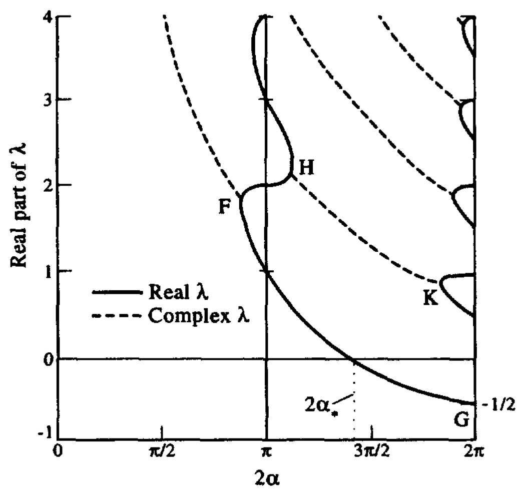

Fig. 2. Roots of $Q(\lambda,\alpha) = (\lambda+1)\sin 2\alpha - \sin 2(\lambda+1)\alpha = 0$. Taken from Ting (1984).

them (see Figs 1 and 2), from which we discover that other more general situations of the paradox have not been studied till now except the two special cases mentioned above, and the researches before all adopt Airy stress function method.

In this paper we employ the complex variable method to study all the cases of the paradox in the range $n \geqslant 0$ and $0 < 2\alpha \leqslant 2\pi$, and the corresponding bounded solutions are obtained. Moreover, the secondary paradox is discovered in the problem, i.e., the initial solution for the paradox is still infinite for some special values of (n, α), this is also resolved here. From the results obtained it can be observed that the stress distribution possesses a $r^n(\ln r)$ term for the paradox and an $r^n(\ln r)^2$ term more for the secondary paradox.

2. BASIC EQUATIONS

In a polar coordinate system (r, θ), let an elastic wedge of angle 2α occupy the region $0 \leqslant r < \infty$, $-\alpha \leqslant \theta \leqslant \alpha$, when the wedge is loaded by tractions proportional to $r^n (n \geqslant 0)$ on the surfaces $\theta = \pm\alpha$, the boundary conditions are

$$\sigma_{\theta\theta}(r,\alpha) = pr^n, \quad \sigma_{r\theta}(r,\alpha) = qr^n$$
$$\sigma_{\theta\theta}(r,-\alpha) = p_2 r^n, \quad \sigma_{r\theta}(r,-\alpha) = q_2 r^n \tag{1}$$

where p, q and p_2, q_2 are real constants.

With regard to (1), we can consider the loadings which are symmetric and antisymmetric separately because an arbitrary loading can always be decomposed into a symmetric and an antisymmetric loading, i.e.,

$$p-p_2 = 0, \quad q+q_2 = 0 \quad \text{for symmetric deformations} \tag{2a}$$
$$p+p_2 = 0, \quad q-q_2 = 0 \quad \text{for antisymmetric deformations} \tag{2b}$$

In this paper the complex variable method is adopted, and the following formulas can be written (Muskhelishvili, 1953)

$$\sigma_{rr} - i\sigma_{r\theta} = \Phi(z) + \overline{\Phi(z)} - e^{2i\theta}[\bar{z}\Phi'(z) + \Psi(z)] \tag{3}$$

$$\sigma_{\theta\theta} + i\sigma_{r\theta} = \Phi(z) + \overline{\Phi(z)} + e^{2i\theta}[\bar{z}\Phi'(z) + \Psi(z)] \tag{4}$$

where $\Phi(z)$ and $\Psi(z)$ are two analytic functions on the complex plane ($z = re^{i\theta}$), $\overline{\Phi(z)}$ is the complex conjugate of $\Phi(z)$.

The boundary conditions (1) can be rewritten in the form of complex as

$$\theta = \alpha : \sigma_{\theta\theta} + i\sigma_{r\theta} = (p+iq)r^n$$
$$\theta = -\alpha : \sigma_{\theta\theta} + i\sigma_{r\theta} = (p_2+iq_2)r^n \tag{5}$$

Based on (4), the simple and evident expressions for $\Phi(z)$ and $\Psi(z)$ which can make the boundary conditions (5) being satisfied are z^n, we assume

$$\Phi(z) = Az^n + Ez^n \ln z + Gz^n(\ln z)^2$$
$$\Psi(z) = Bz^n + Fz^n \ln z + Hz^n(\ln z)^2 \tag{6}$$

where A, B, E, F, G and H are complex constants.

Substituting (6) into (4), then applying the boundary conditions (5), we have

$$(n+1)G\,e^{in\alpha} + \bar{G}\,e^{-in\alpha} + H\,e^{i(n+2)\alpha} = 0$$
$$(n+1)G\,e^{-in\alpha} + \bar{G}\,e^{in\alpha} + H\,e^{-i(n+2)\alpha} = 0$$
$$(n+1)E\,e^{in\alpha} + \bar{E}\,e^{-in\alpha} + F\,e^{i(n+2)\alpha} + 2G[1+i(n+1)\alpha]\,e^{in\alpha} - 2i\bar{G}\alpha\,e^{-in\alpha} + 2iH\alpha\,e^{i(n+2)\alpha} = 0$$
$$(n+1)E\,e^{-in\alpha} + \bar{E}\,e^{in\alpha} + F\,e^{-i(n+2)\alpha} + 2G[1-i(n+1)\alpha]\,e^{-in\alpha} + 2i\bar{G}\alpha\,e^{in\alpha} - 2iH\alpha\,e^{-i(n+2)\alpha} = 0$$
$$(n+1)A\,e^{in\alpha} + \bar{A}\,e^{-in\alpha} + B\,e^{i(n+2)\alpha} + E[1+i(n+1)\alpha]\,e^{in\alpha} - i\bar{E}\alpha\,e^{-in\alpha}$$
$$+ iF\alpha\,e^{i(n+2)\alpha} + G[2i\alpha - (n+1)\alpha^2]\,e^{in\alpha} - \bar{G}\alpha^2\,e^{-in\alpha} - H\alpha^2\,e^{i(n+2)\alpha} = p+iq$$
$$(n+1)A\,e^{-in\alpha} + \bar{A}\,e^{in\alpha} + B\,e^{-i(n+2)\alpha} + E[1-i(n+1)\alpha]\,e^{-in\alpha} + i\bar{E}\alpha\,e^{in\alpha}$$
$$- iF\alpha\,e^{-i(n+2)\alpha} + G[-2i\alpha - (n+1)\alpha^2]\,e^{-in\alpha} - \bar{G}\alpha^2\,e^{in\alpha} - H\alpha^2\,e^{-i(n+2)\alpha} = p_2+iq_2 \tag{7}$$

Equation (7) can be further simplified, for instance, the real part and the imaginary part of the unknown complex constants can be solved separately, thus we obtain the following two sets of equations

$$(G+\bar{G})P(n,\alpha) = 0$$

$$H+\bar{H}+(G+\bar{G})R(n,\alpha) = 0$$

$$(E+\bar{E})P(n,\alpha)+2(G+\bar{G})V(n,\alpha)-4\alpha(H+\bar{H}) = 0$$

$$F+\bar{F}+(E+\bar{E})R(n,\alpha)+2(G+\bar{G})W(n,\alpha) = 0$$

$$(A+\bar{A})P(n,\alpha)+(E+\bar{E})M(n,\alpha)+4(G+\bar{G})(n+1)\alpha^2\sin 2\alpha = \begin{cases} 2X(p,q,n,\alpha) & \text{for symmetric deformations} \\ 0 & \text{for antisymmetric deformations} \end{cases}$$

$$B+\bar{B}+(A+\bar{A})R(n,\alpha)+(E+\bar{E})[W(n,\alpha)-\alpha P(n,\alpha)]+2(G+\bar{G})\alpha\sin 2\alpha = \begin{cases} 2Y(p,q,n,\alpha) & \text{for symmetric deformations} \\ 0 & \text{for antisymmetric deformations} \end{cases} \tag{8a}$$

$$(G-\bar{G})Q(n,\alpha) = 0$$

$$H-\bar{H}+(G-\bar{G})S(n,\alpha) = 0$$

$$(E-\bar{E})Q(n,\alpha)+2(G-\bar{G})V(n,\alpha)-4\alpha(H-\bar{H}) = 0$$

$$F-\bar{F}+(E-\bar{E})S(n,\alpha)+2(G-\bar{G})W(n,\alpha) = 0$$

$$(A-\bar{A})Q(n,\alpha)+(E-\bar{E})N(n,\alpha)+4(G-\bar{G})(n+1)\alpha^2\sin 2\alpha = \begin{cases} 0 & \text{for symmetric deformations} \\ 2iY(p,q,n,\alpha) & \text{for antisymmetric deformations} \end{cases}$$

$$B-\bar{B}+(A-\bar{A})S(n,\alpha)+(E-\bar{E})[W(n,\alpha)-\alpha Q(n,\alpha)]+2(G-\bar{G})\alpha\sin 2\alpha = \begin{cases} 0 & \text{for symmetric deformations} \\ -2iX(p,q,n,\alpha) & \text{for antisymmetric deformations} \end{cases} \tag{8b}$$

where

$$\begin{aligned} P(n,\alpha) &= (n+1)\sin 2\alpha+\sin 2(n+1)\alpha, & Q(n,\alpha) &= (n+1)\sin 2\alpha-\sin 2(n+1)\alpha \\ R(n,\alpha) &= (n+1)\cos 2\alpha+\cos 2(n+1)\alpha, & S(n,\alpha) &= (n+1)\cos 2\alpha-\cos 2(n+1)\alpha \\ V(n,\alpha) &= \sin 2\alpha-2(n+1)\alpha\cos 2\alpha, & W(n,\alpha) &= \cos 2\alpha+2(n+1)\alpha\sin 2\alpha \\ M(n,\alpha) &= \sin 2\alpha+2\alpha\cos 2(n+1)\alpha, & N(n,\alpha) &= \sin 2\alpha-2\alpha\cos 2(n+1)\alpha \end{aligned}$$

$$X(p,q,n,\alpha) = p\sin(n+2)\alpha-q\cos(n+2)\alpha$$

$$Y(p,q,n,\alpha) = p\cos(n+2)\alpha+q\sin(n+2)\alpha \tag{9}$$

The linear eqns (8a) are about $A+\bar{A}$, $B+\bar{B},\ldots,H+\bar{H}$, the coefficient determinant of which is $[P(n,\alpha)]^3$, the linear eqns (8b) are about $A-\bar{A}$, $B-\bar{B},\ldots,H-\bar{H}$, the coefficient determinant of which is $[Q(n,\alpha)]^3$. For symmetric deformations eqns (8a) are nonhomogeneous and eqns (8b) are homogeneous, while for antisymmetric deformations eqns (8a) are homogeneous and eqns (8b) nonhomogeneous.

3. THE PARTICULAR SOLUTIONS OF SYMMETRIC DEFORMATIONS

For symmetric deformations, eqns (8b) are homogeneous, for the sake of simplicity, we take

$$A-\bar{A} = B-\bar{B} = \cdots = H-\bar{H} = 0 \tag{10}$$

therefore only eqns (8a) need to be solved.

3.1. *When* $P(n, \alpha) \neq 0$

Solving eqns (8a) and (10) we get

$$A = \frac{X(p,q,n,\alpha)}{P(n,\alpha)}, \quad B = Y(p,q,n,\alpha) - \frac{X(p,q,n,\alpha)R(n,\alpha)}{P(n,\alpha)}, \quad E = F = G = H = 0$$

substituting them into (6) yields

$$\Phi(z) = \frac{X(p,q,n,\alpha)}{P(n,\alpha)} z^n, \quad \Psi(z) = \left[Y(p,q,n,\alpha) - \frac{X(p,q,n,\alpha)R(n,\alpha)}{P(n,\alpha)} \right] z^n \tag{11}$$

(11) is the classical solution, from which we can reproduce the result of Wang (1986). Especially for $n = 0$, (11) coincides with the result of Dempsey (1981) and of Ting (1984).

3.2. *When* $P(n, \alpha) = 0$, $M(n, \alpha) \neq 0$

In this case the paradox occurs for the classical solution (11). Solving eqns (8a) and (10) we get

$$B = -AR(n,\alpha) + Y(p,q,n,\alpha) - \frac{X(p,q,n,\alpha)W(n,\alpha)}{M(n,\alpha)}$$

$$E = \frac{X(p,q,n,\alpha)}{M(n,\alpha)}, \quad F = -\frac{X(p,q,n,\alpha)R(n,\alpha)}{M(n,\alpha)}, \quad G = H = 0$$

substituting them into (6) yields

$$\Phi(z) = Az^n + \frac{X(p,q,n,\alpha)}{M(n,\alpha)} z^n \ln z$$

$$\Psi(z) = -AR(n,\alpha)z^n + \left[Y(p,q,n,\alpha) - \frac{X(p,q,n,\alpha)W(n,\alpha)}{M(n,\alpha)} \right] z^n - \frac{X(p,q,n,\alpha)R(n,\alpha)}{M(n,\alpha)} z^n \ln z \tag{12}$$

where A is an arbitrary real constant. (12) is the solution for the paradox, which may be called the initial paradox solution.

The analytic functions relevant to the arbitrary real constant A in (12) are

$$\Phi(z) = Az^n, \quad \Psi(z) = -AR(n,\alpha)z^n \tag{13}$$

which provide zero boundary tractions at the boundaries $\theta = \pm\alpha$, thus, (13) is a homogeneous solution.

Substituting (12) into (3) and (4), we discover that the stress distribution contains a $r^n(\ln r)$ term.

For the special case $2\alpha = \pi$, $n = 1, 2, 3, \ldots$ and $2\alpha = 2\pi$, $n = 1/2, 1, 3/2, 2, \ldots$,

$$M(n,\alpha) = \begin{cases} -\pi, & \text{for } 2\alpha = \pi, \quad n = 2, 4, \ldots \\ \pi, & \text{for } 2\alpha = \pi, \quad n = 1, 3, \ldots \\ 2\pi, & \text{for } 2\alpha = 2\pi, \quad n = 1, 2, \ldots \\ -2\pi, & \text{for } 2\alpha = 2\pi, \quad n = 1/2, 3/2, \ldots \end{cases} \tag{14}$$

from (12) we can reproduce the result of Wang (1986) by taking $A = [\alpha Y(p,q,n,\alpha)/M(n,\alpha)]$.

In particular, for the special case $n = 0$, the classical solution (11) breaks down when $2\alpha = \pi$ or 2π, from (12) we also can reproduce the result of Ting (1984) (by taking $A = p/2$) and of Dempsey (1981).

It should be noted that the initial paradox solution (12) is still infinite when $P(n,\alpha) = M(n,\alpha) = 0$, this is the secondary paradox, and the values of (n,α) satisfying $P(n,\alpha) = M(n,\alpha) = 0$ correspond to the specific points on the curve of $P(n,\alpha) = 0$ at which the partial derivative $[\partial P(n,\alpha)/\partial n](=M(n,\alpha))$ vanishes, such as the points F, H and K in Fig. 1.

3.3. *When* $\mathrm{P}(\mathrm{n},\alpha) = \mathrm{M}(\mathrm{n},\alpha) = 0$

Solving eqns (8a) and (10) we get

$$B = -AR(n,\alpha) - EW(n,\alpha) + Y(p,q,n,\alpha) - \frac{X(p,q,n,\alpha)}{2(n+1)\alpha}$$

$$F = -ER(n,\alpha) - \frac{X(p,q,n,\alpha)W(n,\alpha)}{2(n+1)\alpha^2 \sin 2\alpha}$$

$$G = \frac{X(p,q,n,\alpha)}{4(n+1)\alpha^2 \sin 2\alpha}, \quad H = -\frac{X(p,q,n,\alpha)R(n,\alpha)}{4(n+1)\alpha^2 \sin 2\alpha}$$

substituting them into (6) yields

$$\begin{aligned}\Phi(z) &= Az^n + Ez^n \ln z + \frac{X(p,q,n,\alpha)}{4(n+1)\alpha^2 \sin 2\alpha} z^n \ln^2 z \\ \Psi(z) &= -AR(n,\alpha)z^n - EW(n,\alpha)z^n - ER(n,\alpha)z^n \ln z \\ &\quad + \left[Y(p,q,n,\alpha) - \frac{X(p,q,n,\alpha)}{2(n+1)\alpha} \right] z^n - \frac{X(p,q,n,\alpha)W(n,\alpha)}{2(n+1)\alpha^2 \sin 2\alpha} z^n \ln z \\ &\quad - \frac{X(p,q,n,\alpha)R(n,\alpha)}{4(n+1)\alpha^2 \sin 2\alpha} z^n \ln^2 z \end{aligned} \tag{15}$$

where A and E are arbitrary real constants. (15) is just the solution for the secondary paradox, which may be called the secondary paradox solution.

The analytic functions relevant to the arbitrary real constants A and E in (15) are

$$\begin{aligned}\Phi(z) &= Az^n + Ez^n \ln z \\ \Psi(z) &= -AR(n,\alpha)z^n - EW(n,\alpha)z^n - ER(n,\alpha)z^n \ln z \end{aligned} \tag{16}$$

which provide zero boundary tractions on the sides $\theta = \pm\alpha$, thus, (16) is a homogeneous solution.

Substituting (15) into (3) and (4), we discover that the stress distribution contains $r^n(\ln r)$ and $r^n(\ln r)^2$ terms.

Because $\sin 2\alpha \neq 0$ when $P(n,\alpha) = M(n,\alpha) = 0$, the denominator of the solution (15) does not vanish, hence the paradox does not exist for the secondary paradox solution (15).

Now, we have obtained the classical solution (11), the initial paradox solution (12) and the secondary paradox solution (15) for symmetric deformations. However, when $P(n,\alpha) = 0$ and $M(n,\alpha)$ very close to zero, the initial paradox solution (12) is still very large, so is the classical solution (11) when $P(n,\alpha)$ approaches zero, this problem can be solved by constructing modified particular solutions, which will be presented later.

4. THE PARTICULAR SOLUTIONS OF ANTISYMMETRIC DEFORMATIONS

For antisymmetric deformations, eqns (8a) are homogeneous, so we take

$$A+\bar{A} = B+\bar{B} = \cdots = H+\bar{H} = 0 \tag{17}$$

therefore only eqns (8b) need to be solved.

4.1. *When* Q(n, α) ≠ 0

Solving eqns (8b) and (17) we get

$$A = i\frac{Y(p,q,n,\alpha)}{Q(n,\alpha)}, \quad B = -i\left[X(p,q,n,\alpha)+\frac{Y(p,q,n,\alpha)S(n,\alpha)}{Q(n,\alpha)}\right], \quad E = F = G = H = 0$$

substituting them into (6) yields

$$\Phi(z) = i\frac{Y(p,q,n,\alpha)}{Q(n,\alpha)}z^n, \quad \Psi(z) = -i\left[X(p,q,n,\alpha)+\frac{Y(p,q,n,\alpha)S(n,\alpha)}{Q(n,\alpha)}\right]z^n \tag{18}$$

(18) is the classical solution ($n > 0$), from which we can reproduce the result of Wang (1986). For $n = 0$, $Q(n,\alpha) \equiv 0$, this special case is discussed in the following two sections.

4.2. *When* Q(n, α) = 0, N(n, α) ≠ 0

In this case the paradox occurs for the classical solution (18). Solving eqns (8b) and (17) we get

$$B = -AS(n,\alpha)-i\left[X(p,q,n,\alpha)+\frac{Y(p,q,n,\alpha)W(n,\alpha)}{N(n,\alpha)}\right]$$

$$E = i\frac{Y(p,q,n,\alpha)}{N(n,\alpha)}, \quad F = -i\frac{Y(p,q,n,\alpha)S(n,\alpha)}{N(n,\alpha)}, \quad G = H = 0$$

substituting them into (6) yields

$$\Phi(z) = Az^n + i\frac{Y(p,q,n,\alpha)}{N(n,\alpha)}z^n \ln z$$

$$\Psi(z) = -AS(n,\alpha)z^n - i\left[X(p,q,n,\alpha)+\frac{Y(p,q,n,\alpha)W(n,\alpha)}{N(n,\alpha)}\right]z^n - i\frac{Y(p,q,n,\alpha)S(n,\alpha)}{N(n,\alpha)}z^n \ln z \tag{19}$$

where A is an arbitrary imaginary constant. (19) is the solution for the paradox, which can be called the initial paradox solution ($n \geqslant 0$). For the special case $n = 0$, (19) coincides with the result of Dempsey (1981) and of Ting (1984), but they both take it as the classical solution.

The analytic functions relevant to the arbitrary imaginary constant A in (19) are

$$\Phi(z) = Az^n, \quad \Psi(z) = -AS(n,\alpha)z^n \tag{20}$$

which provide zero boundary tractions on the surfaces $\theta = \pm\alpha$, so (20) is a homogeneous solution.

Substituting (19) into (3) and (4), we discover that when $n > 0$ the stress distribution contains a $r^n(\ln r)$ term, while when $n = 0$ the $(\ln r)$ term disappears.

For the special case $2\alpha = \pi$, $n = 1, 2, 3, \ldots$ and $2\alpha = 2\pi$, $n = 1/2, 1, 3/2, 2, \ldots,$

$$N(n,\alpha) = \begin{cases} \pi, & \text{for } 2\alpha = \pi, \quad n = 2,4,\dots \\ -\pi, & \text{for } 2\alpha = \pi, \quad n = 1,3,\dots \\ -2\pi, & \text{for } 2\alpha = 2\pi, \quad n = 1,2,\dots \\ 2\pi, & \text{for } 2\alpha = 2\pi, \quad n = 1/2, 3/2,\dots \end{cases} \tag{21}$$

from (19) we can reproduce the result of Wang (1986) by taking $A = -i[\alpha X(p,q,n,\alpha)/N(n,\alpha)]$.

It should be noted that the initial paradox solution (19) is still infinite when $Q(n,\alpha) = N(n,\alpha) = 0$, this is the secondary paradox. For $n > 0$, the values of (n,α) satisfying $Q(n,\alpha) = N(n,\alpha) = 0$ correspond to the specific points on the curve of $Q(n,\alpha) = 0$ at which the partial derivative $[\partial Q(n,\alpha)/\partial n](=N(n,\alpha))$ vanishes, such as the points F, H and K in Fig. 2; for $n = 0$, the values of (n,α) satisfying $Q(n,\alpha) = N(n,\alpha) = 0$ are $(0,\alpha_*)$.

4.3. *When* Q(n, α) = N(n, α) = 0

Solving eqns (8b) and (17) we get

$$B = -AS(n,\alpha) - EW(n,\alpha) - i\left[X(p,q,n,\alpha) + \frac{Y(p,q,n,\alpha)}{2(n+1)\alpha}\right]$$

$$F = -ES(n,\alpha) - i\frac{Y(p,q,n,\alpha)W(n,\alpha)}{2(n+1)\alpha^2 \sin 2\alpha}$$

$$G = i\frac{Y(p,q,n,\alpha)}{4(n+1)\alpha^2 \sin 2\alpha}, \quad H = -i\frac{Y(p,q,n,\alpha)S(n,\alpha)}{4(n+1)\alpha^2 \sin 2\alpha}$$

substituting them into (6) yields

$$\Phi(z) = Az^n + Ez^n \ln z + i\frac{Y(p,q,n,\alpha)}{4(n+1)\alpha^2 \sin 2\alpha} z^n \ln^2 z$$

$$\begin{aligned} \Psi(z) = &-AS(n,\alpha)z^n - EW(n,\alpha)z^n - ES(n,\alpha)z^n \ln z \\ &-i\left[X(p,q,n,\alpha) + \frac{Y(p,q,n,\alpha)}{2(n+1)\alpha}\right]z^n - i\frac{Y(p,q,n,\alpha)W(n,\alpha)}{2(n+1)\alpha^2 \sin 2\alpha} z^n \ln z \\ &-i\frac{Y(p,q,n,\alpha)S(n,\alpha)}{4(n+1)\alpha^2 \sin 2\alpha} z^n \ln^2 z \end{aligned} \tag{22}$$

where A and E are arbitrary imaginary constants. (22) is just the solution for the secondary paradox, which can be called the secondary paradox solution. For the special case $n = 0$, (22) coincides with the result of Ting (1984) (by taking $E = -ip/2\alpha$) and of Dempsey (1981), but they both take it as the (initial) paradox solution.

The analytic functions relevant to the arbitrary imaginary constants A and E in (22) are

$$\Phi(z) = Az^n + Ez^n \ln z,$$

$$\Psi(z) = -AS(n,\alpha)z^n - EW(n,\alpha)z^n - ES(n,\alpha)z^n \ln z \tag{23}$$

which provide zero boundary tractions on the sides $\theta = \pm\alpha$, thus, (23) is a homogeneous solution.

Substituting (22) into (3) and (4), we discover that when $n > 0$ the stress distribution contains $r^n(\ln r)$ and $r^n(\ln r)^2$ terms, while when $n = 0$ the $(\ln r)^2$ term disappears.

Because $\sin 2\alpha \neq 0$ when $Q(n,\alpha) = N(n,\alpha) = 0$, the denominator of the solution (22) does not vanish, hence the paradox does not exist for the secondary paradox solution (22).

Now, we have obtained the classical solution (18), the initial paradox solution (19) and the secondary paradox solution (22) for antisymmetric deformations. However, when $Q(n, \alpha) = 0$ and $N(n, \alpha)$ very close to zero, the initial paradox solution (19) is still very large, so is the classical solution (18) when $Q(n, \alpha)$ approaches zero, this problem can be solved by constructing modified particular solutions, which will be presented later.

5. HOMOGENEOUS SOLUTIONS

Setting $p = q = p_2 = q_2 = 0$ in (5), the homogeneous boundary conditions are

$$\theta = \pm\alpha, \quad \sigma_{\theta\theta} + i\sigma_{r\theta} = 0 \tag{24}$$

We assume that the homogeneous solution is

$$\Phi(z) = Cz^{\eta} + Jz^{\bar{\eta}}, \quad \Psi(z) = Dz^{\eta} + Kz^{\bar{\eta}} \tag{25}$$

where $\bar{\eta}$ is the complex conjugate of η, C, D, J and K are complex constants.

Substituting (25) into (4), then applying the homogeneous boundary conditions (24), one obtains

$$\begin{aligned} (\eta+1)C\,e^{i\eta\alpha} + \bar{J}\,e^{-i\eta\alpha} + D\,e^{i(\eta+2)\alpha} &= 0 \\ (\eta+1)C\,e^{-i\eta\alpha} + \bar{J}\,e^{i\eta\alpha} + D\,e^{-i(\eta+2)\alpha} &= 0 \\ \bar{C}\,e^{-i\bar{\eta}\alpha} + (\bar{\eta}+1)J\,e^{i\bar{\eta}\alpha} + K\,e^{i(\bar{\eta}+2)\alpha} &= 0 \\ \bar{C}\,e^{i\bar{\eta}\alpha} + (\bar{\eta}+1)J\,e^{-i\bar{\eta}\alpha} + K\,e^{-i(\bar{\eta}+2)\alpha} &= 0 \end{aligned} \tag{26}$$

Equation (26) can be further simplified as

$$\begin{aligned} (\eta+1)C(e^{-2i\alpha} - e^{2i\alpha}) + \bar{J}[e^{-i(2\eta+2)\alpha} - e^{i(2\eta+2)\alpha}] &= 0 \\ C[e^{-i(2\eta+2)\alpha} - e^{i(2\eta+2)\alpha}] + (\eta+1)\bar{J}(e^{-2i\alpha} - e^{2i\alpha}) &= 0 \\ (\eta+1)C(e^{-2i\alpha} + e^{2i\alpha}) + \bar{J}[e^{-i(2\eta+2)\alpha} + e^{i(2\eta+2)\alpha}] + 2D &= 0 \\ \bar{C}[e^{-i(2\bar{\eta}+2)\alpha} + e^{i(2\bar{\eta}+2)\alpha}] + (\bar{\eta}+1)J(e^{-2i\alpha} + e^{2i\alpha}) + 2K &= 0 \end{aligned} \tag{27}$$

From the former two of eqn (27), it is easy to know that only when η satisfies $P(\eta, \alpha) = 0$ or $Q(\eta, \alpha) = 0$, a non-trivial solution for C and J exists, and from the latter two of eqn (27) D and K can be obtained. Therefore the following two sets of homogeneous solutions are derived:

(1) when $P(\eta, \alpha) = 0$

$$\Phi(z) = Cz^{\eta} + \bar{C}z^{\bar{\eta}}, \quad \Psi(z) = -CR(\eta, \alpha)z^{\eta} - \bar{C}R(\bar{\eta}, \alpha)z^{\bar{\eta}} \tag{28a}$$

where C is an arbitrary complex constant.

Especially, if $\eta = \bar{\eta} = \lambda$ where λ is a real number, (28a) becomes

$$\Phi(z) = Lz^{\lambda}, \quad \Psi(z) = -LR(\lambda, \alpha)z^{\lambda} \tag{28b}$$

where $L = C + \bar{C}$ is an arbitrary real constant.

(2) when $Q(\eta,\alpha) = 0$,

$$\Phi(z) = Cz^{\eta} - \bar{C}z^{\bar{\eta}}, \quad \Psi(z) = -CS(\eta,\alpha)z^{\eta} + \bar{C}S(\bar{\eta},\alpha)z^{\bar{\eta}} \tag{29a}$$

where C is an arbitrary complex constant.

Especially, if $\eta = \bar{\eta} = \lambda$ where λ is a real number, (29a) becomes

$$\Phi(z) = Lz^{\lambda}, \quad \Psi(z) = -LS(\lambda,\alpha)z^{\lambda} \tag{29b}$$

where $L = C - \bar{C}$ is an arbitrary imaginary constant.

By properly choosing the above homogeneous solutions and the arbitrary constants in them, then superimposing the homogeneous solutions to the initial paradox solutions or the classical solutions, we can construct the modified particular solutions which remain bounded as the denominators of them approach zero.

6. THE MODIFIED PARTICULAR SOLUTIONS OF SYMMETRIC DEFORMATIONS

6.1. *The modified paradox solution* (*when* $\mathrm{P}(n,\alpha) = 0$ *and* $\mathrm{M}(n,\alpha)$ *very close to zero*)

Let (n^*, α^*) denote the roots of the equations $P(n,\alpha) = M(n,\alpha) = 0$, it corresponds to the points such as F, H and K in Fig. 1. For the initial paradox solution (12), when $P(n,\alpha) = 0$ and $M(n,\alpha)$ very close to zero, (n,α) is certainly near (n^*,α^*) and on the curve of $P(\lambda,\alpha) = 0$, thus it is evident that on the curve of $P(\lambda,\alpha) = 0$ there exists another point (n',α) which is also near (n^*,α^*), so we can construct the following homogeneous solution from (28b).

$$\begin{aligned}
\Phi(z) &= \frac{L}{n'-n}(z^{n'} - z^{n}) = Lz^{n}\cdot\frac{z^{n'-n}-1}{n'-n} \\
\Psi(z) &= -\frac{L}{n'-n}[R(n',\alpha)z^{n'} - R(n,\alpha)z^{n}] \\
&= -Lz^{n}\cdot\left[R(n',\alpha)\frac{z^{n'-n}-1}{n'-n} + \frac{R(n',\alpha)-R(n,\alpha)}{n'-n}\right]
\end{aligned} \tag{30}$$

where L is an arbitrary real constant.

Choosing $L = -[X(p,q,n,\alpha)/M(n,\alpha)]$ in (30) and superimposing it to the initial paradox solution (12) (by taking $A = 0$), we obtain a new particular solution for the paradox

$$\begin{aligned}
\Phi(z) &= X(p,q,n,\alpha)\frac{\ln z - \dfrac{z^{n'-n}-1}{n'-n}}{M(n,\alpha)}z^{n} \\
\Psi(z) &= Y(p,q,n,\alpha)z^{n} + X(p,q,n,\alpha) \\
&\quad\cdot\left[\frac{\dfrac{R(n',\alpha)-R(n,\alpha)}{n'-n} - W(n,\alpha)}{M(n,\alpha)} + \frac{R(n',\alpha)\dfrac{z^{n'-n}-1}{n'-n} - R(n,\alpha)\ln z}{M(n,\alpha)}\right]z^{n}
\end{aligned} \tag{31}$$

as $2\alpha \to 2\alpha^*$, n and $n'n^*$, $[R((n',\alpha) - R(n,\alpha))/(n'-n)] \to W(n^*,\alpha)$, $(z^{n'-n}-1)/(n'-n) \to \ln z$, therefore all fractions in (31) become the indeterminate mode of 0/0, and it can be shown that

$$\lim_{\substack{2\alpha\to 2\alpha^* \\ (n,n'\to n^*)}} \frac{dn'}{d\alpha} \bigg/ \frac{dn}{d\alpha} = -1. \tag{32}$$

By applying L'Hospital's rule and using eqn (32), it can be proved that the limitation of (31) is the secondary paradox solution (15) (by replacing n, α with n^*, α^* and taking $A = E = 0$). Hence (31) is bounded when $P(n, \alpha) = 0$ and $M(n, \alpha)$ very close to zero, which can be called the modified paradox solution.

According to the discussion above, when $P(n, \alpha) = 0$, the bounded paradox solutions of symmetric deformations include:

(1) the initial paradox solution (12), which is suitable for the case $M(n, \alpha) \neq 0$ and not close to zero;
(2) the secondary paradox solution (15), which is suitable for the case $M(n, \alpha) = 0$;
(3) the modified paradox solution (31), which is suitable for the case $M(n, \alpha)$ very close to zero.

6.2. *The modified classical solutions (when* $P(n, \alpha)$ *very close to zero)*

For the classical solution (11), when $P(n, \alpha)$ very close to zero, the point (n, α) is certainly in the neighbouring region of the curve of $P(\lambda, \alpha) = 0$, and there are two possible situations: one is that there exists a wedge angle $2\alpha_0$ which is near 2α and satisfies $P(n, \alpha_0) = 0$, the other is that there exists a constant $n_0(\geqslant 0)$ which is near n and satisfies $P(n_0, \alpha) = 0$. Of course, for a lot of (n, α), there exists not only $2\alpha_0$ but also n_0, but we mainly study the first situation, i.e., we construct the modified solutions which are bounded as 2α approaches $2\alpha_0$ while the loading remains permanent (n is fixed); for the second situation, we shall construct the modified solutions which are bounded as n approaches n_0 while 2α remains fixed. It can be observed easily from Fig. 1 that when 2α equal or very close to 2π, and n very close to but larger than the semi-integer or n very close to but smaller than the integer, the point (n, α) only belongs to the second situation.

6.2.1. *When* $M(n, \alpha_0) \neq 0$ *and not close to zero.* In this case (n, α_0) is not near (n^*, α^*), and it can be seen from Fig. 1 that on the curve of $P(\lambda, \alpha) = 0$ there always exists a point (λ, α) near (n, α) as 2α approaches $2\alpha_0$ while n is fixed, hence the homogeneous solution (28b) exists, in which we choose $L = -[X(p, q, \lambda, \alpha)/P(n, \alpha)]$ and superimpose it to the classical solution (11), the result is

$$\Phi(z) = \frac{X(p,q,n,\alpha)z^n - X(p,q,\lambda,\alpha)z^\lambda}{P(n,\alpha)}$$

$$\Psi(z) = Y(p,q,n,\alpha)z^n - \frac{X(p,q,n,\alpha)R(n,\alpha)z^n - X(p,q,\lambda,\alpha)R(\lambda,\alpha)z^\lambda}{P(n,\alpha)} \tag{33}$$

as $2\alpha \to 2\alpha_0$, $\lambda \to n$, the fractions in (33) become the indeterminate mode of 0/0, and it can be demonstrated that

$$\lim_{\substack{2\alpha\to 2\alpha_0 \\ (\lambda\to n)}} \frac{d\lambda}{d\alpha} \bigg/ \frac{dP(n,\alpha)}{d\alpha} = -\frac{1}{M(n,\alpha_0)} \tag{34}$$

By applying L'Hospital's rule and using eqn (34), it is shown that the limitation of (33) is the initial paradox solution (12) (by replacing α with α_0 and taking $A = [\alpha_0 Y(p, q, n, \alpha_0)/M(n, \alpha_0)]$), thus, (33) is bounded, which can be called the modified classical solution.

For the special case $2\alpha_0 = \pi$, $n = 1, 2, 3, \ldots$ and $2\alpha_0 = 2\pi$, $n = 1/2, 1, 3/2, 2, \ldots$, (33) is coincident with the result of Wang (1986).

For the special case $n = 0$, $2\alpha_0 = \pi$ or 2π, if we choose $L = [(q\cos(\lambda+2)\alpha)/2\sin 2\alpha]$ in the homogeneous solution (28b) and superimpose it to the classical solution (11), we can get the result of Ting (1984).

6.2.2. *When* n = n*, $2\alpha_0 = 2\alpha^*$. In this case $M(n, \alpha_0) = 0$, we construct the modified solutions under the following two different circumstances:

(a) As 2α approaches $2\alpha^*$ in the direction of turning towards π or 2π, from Fig. 1 it is observed that there exists a pair of complex numbers η and $\bar{\eta}$ which are near n^* and satisfy $P(\eta, \alpha) = 0$ and $P(\bar{\eta}, \alpha) = 0$, hence we can construct the homogeneous solution from (28a) as below

$$\Phi(z) = L_1 \frac{z^\eta + z^{\bar{\eta}}}{2} + L_2 \frac{z^\eta - z^{\bar{\eta}}}{\eta - \bar{\eta}}$$

$$\Psi(z) = -L_1 \frac{R(\eta, \alpha)z^\eta + R(\bar{\eta}, \alpha)z^{\bar{\eta}}}{2} - L_2 \frac{R(\eta, \alpha)z^\eta - R(\bar{\eta}, \alpha)z^{\bar{\eta}}}{\eta - \bar{\eta}} \tag{35}$$

where L_1 and L_2 are arbitrary real constants.

We choose

$$L_1 = -\frac{X(p, q, n^*, \alpha)}{P(n^*, \alpha)} \quad \text{and} \quad L_2 = \frac{X(p, q, n^*, \alpha)\left(\dfrac{\eta+\bar{\eta}}{2} - n^*\right)}{p(n^*, \alpha)}$$

in (35) and superimpose it to the classical solution (11), the result is

$$\Phi(z) = \frac{1}{2}X(p, q, n^*, \alpha)\frac{(z^{n^*} - z^\eta) + (\eta - n^*)z^\eta \dfrac{z^{\bar{\eta}-\eta} - 1}{\bar{\eta} - \eta}}{P(n^*, \alpha)} + \frac{1}{2}X(p, q, n^*, \alpha)\frac{(z^{n^*} - z^{\bar{\eta}}) + (\bar{\eta} - n^*)z^{\bar{\eta}} \dfrac{z^{\eta-\bar{\eta}} - 1}{\eta - \bar{\eta}}}{P(n^*, \alpha)} \tag{36a}$$

$$\Psi(z) = Y(p, q, n^*, \alpha)z^{n^*} - \frac{1}{2}X(p, q, n^*, \alpha) \cdot \frac{[R(n^*, \alpha)z^{n^*} - R(\eta, \alpha)z^\eta] + (\eta - n^*)z^\eta\left[R(\bar{\eta}, \alpha)\dfrac{z^{\bar{\eta}-\eta} - 1}{\bar{\eta} - \eta} + \dfrac{R(\bar{\eta}, \alpha) - R(\eta, \alpha)}{\bar{\eta} - \eta}\right]}{P(n^*, \alpha)} - \frac{1}{2}X(p, q, n^*, \alpha) \cdot \frac{[R(n^*, \alpha)z^{n^*} - R(\bar{\eta}, \alpha)z^{\bar{\eta}}] + (\bar{\eta} - n^*)z^{\bar{\eta}}\left[R(\eta, \alpha)\dfrac{z^{\eta-\bar{\eta}} - 1}{\eta - \bar{\eta}} + \dfrac{R(\eta, \alpha) - R(\bar{\eta}, \alpha)}{\eta - \bar{\eta}}\right]}{P(n^*, \alpha)} \tag{36b}$$

(b) As 2α approaches $2\alpha^*$ in the direction of deviating from π or 2π, from Fig. 1 it is observed that there exists two real numbers λ and λ' which are near n^* and satisfy $P(\lambda, \alpha) = 0$ and $P(\lambda', \alpha) = 0$, hence we can construct the homogeneous solution from (28b) as below

$$\Phi(z) = L_1 z^{\lambda} + L_2 \frac{z^{\lambda'} - z^{\lambda}}{\lambda' - \lambda}$$

$$\Psi(z) = -L_1 R(\lambda,\alpha) z^{\lambda} - L_2 \frac{R(\lambda',\alpha) z^{\lambda'} - R(\lambda,\alpha) z^{\lambda}}{\lambda' - \lambda} \tag{37}$$

where L_1 and L_2 are arbitrary real constants.

We choose

$$L_1 = \frac{X(p,q,n^*,\alpha)}{P(n^*,\alpha)} \quad \text{and} \quad L_2 = \frac{X(p,q,n^*,\alpha)(\lambda - n^*)}{P(n^*,\alpha)}$$

in (37) and superimpose it to the classical solution (11), the result is

$$\Phi(z) = X(p,q,n^*,\alpha) \frac{(z^{n^*} - z^{\lambda}) + (\lambda - n^*) z^{\lambda} \dfrac{z^{\lambda'-\lambda} - 1}{\lambda' - \lambda}}{P(n^*,\alpha)}$$

$$\Psi(z) = Y(p,q,n^*,\alpha) z^{n^*} - X(p,q,n^*,\alpha)$$

$$\cdot \frac{[R(n^*,\alpha) z^{n^*} - R(\lambda,\alpha) z^{\lambda}] + (\lambda - n^*) z^{\lambda} \left[R(\lambda',\alpha) \dfrac{z^{\lambda'-\lambda} - 1}{\lambda' - \lambda} + \dfrac{R(\lambda',\alpha) - R(\lambda,\alpha)}{\lambda' - \lambda} \right]}{P(n^*,\alpha)} \tag{38}$$

as $2\alpha \to 2\alpha^*$, for (36) η and $\bar{\eta} \to n^*$, and it can be shown that $(\mathrm{d}\bar{\eta}/\mathrm{d}\alpha)/(\mathrm{d}\eta/\mathrm{d}\alpha) \to -1$; for (38) λ and $\lambda' \to n^*$, and it can be shown from (32) that $(\mathrm{d}\lambda'/\mathrm{d}\alpha)/(\mathrm{d}\lambda/\mathrm{d}\alpha) \to -1$. By applying L'Hospital's rule twice in succession, it is shown that the limitations of (36) and (38) are both the secondary paradox solution (15) (by replacing n, α with n^*, α^* and taking $A = E = 0$), hence the modified classical solutions (36) and (38) are bounded.

6.2.3. *When* M(n, α_0) *very close to zero.* In this case (n, α_0) is near (n^*, α^*), and it can be observed from Fig. 1 that on the curve of $P(\lambda, \alpha) = 0$ there exists another point (n', α_0) which is also near (n^*, α^*).

(a) As 2α approaches $2\alpha_0$ in the direction of turning towards π or 2π, the modified classical solutions can be constructed under the following three different circumstances:

(1) If 2α is not so close to $2\alpha_0$ as $2\alpha^*$, there exists a pair of complex numbers η and $\bar{\eta}$ which are near n^* and satisfy $P(\eta, \alpha) = 0$ and $P(\bar{\eta}, \alpha) = 0$, using the same method as in 6.2.2(a) we obtain

$$\Phi(z) = X(p,q,n,\alpha) \frac{\left(z^n - \dfrac{z^{\eta} + z^{\bar{\eta}}}{2} \right) + \left(\dfrac{\eta + \bar{\eta}}{2} - n \right) z^{\eta} \dfrac{z^{\bar{\eta}-\eta} - 1}{\bar{\eta} - \eta}}{P(n,\alpha)}$$

$$\Psi(z) = Y(p,q,n,\alpha) z^n - X(p,q,n,\alpha) \frac{\left[R(n,\alpha) z^n - \dfrac{R(\eta,\alpha) z^{\eta} + R(\bar{\eta},\alpha) z^{\bar{\eta}}}{2} \right]}{P(n,\alpha)}$$

$$- X(p,q,n,\alpha) \frac{\left(\dfrac{\eta + \bar{\eta}}{2} - n \right) z^{\eta} \left[R(\bar{\eta},\alpha) \dfrac{z^{\bar{\eta}-\eta} - 1}{\bar{\eta} - \eta} + \dfrac{R(\bar{\eta},\alpha) - R(\eta,\alpha)}{\bar{\eta} - \eta} \right]}{P(n,\alpha)} \tag{39a}$$

(2) If $2\alpha = 2\alpha^*$, we take the limitation of (39a) as $2\alpha \to 2\alpha^*$ (thus, η and $\bar{\eta} \to n^*$), i.e.,

$$\Phi(z) = X(p,q,n,\alpha^*)\frac{(z^n - z^{n^*}) + (n^* - n)z^{n^*}\ln z}{P(n,\alpha^*)}$$

$$\Psi(z) = Y(p,q,n,\alpha^*)z^n - X(p,q,n,\alpha^*)$$

$$\cdot\frac{[R(n,\alpha^*)z^n - R(n^*,\alpha^*)z^{n^*}] + (n^*-n)z^{n^*}[R(n^*,\alpha^*)\ln z + W(n^*,\alpha^*)]}{P(n,\alpha^*)} \quad (39b)$$

(3) If 2α is closer to $2\alpha_0$ than $2\alpha^*$, there exists two real numbers λ and λ' which are near n and n', respectively, and satisfy $P(\lambda,\alpha) = 0$ and $P(\lambda',\alpha) = 0$, using the same method as in 6.2.2(b), we obtain

$$\Phi(z) = X(p,q,n,\alpha)\frac{(z^n - z^\lambda) + (\lambda - n)z^\lambda \dfrac{z^{\lambda'-\lambda}-1}{\lambda'-\lambda}}{P(n,\alpha)}$$

$$\Psi(z) = Y(p,q,n,\alpha)z^n - X(p,q,n,\alpha)$$

$$\cdot\frac{[R(n,\alpha)z^n - R(\lambda,\alpha)z^\lambda] + (\lambda-n)z^\lambda\left[R(\lambda',\alpha)\dfrac{z^{\lambda'-\lambda}-1}{\lambda'-\lambda} + \dfrac{R(\lambda',\alpha)-R(\lambda,\alpha)}{\lambda'-\lambda}\right]}{P(n,\alpha)} \quad (39c)$$

it is noticed that the limitation of (39c) as $2\alpha \to 2\alpha^*$ (thus, λ and $\lambda' \to n^*$) is also (39b), therefore the modified solutions (39a–c) connect continuously at $2\alpha = 2\alpha^*$.

(b) As 2α approaches $2\alpha_0$ in the direction of deviation from π or 2π, we can construct the modified solution (39c).

As $2\alpha \to 2\alpha_0$, $\lambda \to n$ and $\lambda' \to n'$, by applying L'Hospital's rule and eqn (34), it is shown that the limitation of (39c) is the modified paradox solution (31) (by replacing α with α_0), hence the modified classical solution (39) is bounded.

6.2.4. *When 2α equal or very close to 2π,* n *very close to but larger than the semi-integer or* n *very close to but smaller than the integer.* In this case it can be observed from Fig. 1 that on the curve of $P(\lambda,\alpha) = 0$ there always exists a point (n_0,α) $(n_0 > 0)$ near (n,α), moreover, (n_0,α) is not near (n^*,α^*), thus $M(n_0,\alpha) \neq 0$ and not close to zero. We take $\lambda = n_0$ and choose $L = -[X(p,q,n,\alpha)/P(n,\alpha)]$ in the homogeneous solution (28b), then superimpose it to (11), the result is

$$\Phi(z) = X(p,q,n,\alpha)\frac{z^n - z^{n_0}}{P(n,\alpha)}$$

$$\Psi(z) = Y(p,q,n,\alpha)z^n - X(p,q,n,\alpha)\frac{R(n,\alpha)z^n - R(n_0,\alpha)z^{n_0}}{P(n,\alpha)} \quad (40)$$

as $n \to n_0$, the limitation of (40) is the initial paradox solution (12) (by replacing n with n_0 and taking $A = 0$), hence the modified classical solution (40) is bounded.

Now, for symmetric deformations, the modified particular solutions which are bounded as the denominators of them approach zero have been constructed completely.

7. THE MODIFIED PARTICULAR SOLUTIONS OF ANTISYMMETRIC DEFORMATIONS

7.1. *The modified paradox solution* (*when* $Q(n,\alpha) = 0$ *and* $N(n,\alpha)$ *very close to zero*)

Let (n^*,α^*) denote the roots of the equations $Q(n,\alpha) = N(n,\alpha) = 0$, it corresponds to the points such as F, H, K and $(0,\alpha_*)$ in Fig. 2. For the initial paradox solution (19), when

$Q(n,\alpha)=0$ and $N(n,\alpha)$ very close to zero, (n,α) is certainly near (n^*,α^*) and on the curve of $Q(\lambda,\alpha)=0$ (including $\lambda=0$), thus, it is evident that on the curve of $Q(\lambda,\alpha)=0$ there exists another point (n',α) which is also near (n^*,α^*), so we can construct the following homogeneous solution from (29b)

$$\Phi(z)=\frac{L}{n'-n}(z^{n'}-z^n)=L\cdot\frac{z^{n'}-z^n}{n'-n}$$

$$\Psi(z)=-\frac{L}{n'-n}[S(n',\alpha)z^{n'}-S(n,\alpha)z^n]$$

$$=-L\cdot\left[S(n',\alpha)\frac{z^{n'}-z^n}{n'-n}+\frac{S(n',\alpha)-S(n,\alpha)}{n'-n}z^n\right] \tag{41}$$

where L is an arbitrary imaginary constant.

Choosing $L=-i[Y(p,q,n,\alpha)/N(n,\alpha)]$ in (41) and superimposing it to the initial paradox solution (19) (by taking $A=0$), we obtain a new particular solution for the paradox

$$\Phi(z)=iY(p,q,n,\alpha)\frac{z^n\ln z-\dfrac{z^{n'}-z^n}{n'-n}}{N(n,\alpha)} \tag{42a}$$

$$\Psi(z)=-iX(p,q,n,\alpha)z^n+iY(p,q,n,\alpha)$$

$$\cdot\left[\frac{\dfrac{S(n',\alpha)-S(n,\alpha)}{n'-n}-W(n,\alpha)}{N(n,\alpha)}z^n+\frac{S(n',\alpha)\dfrac{z^{n'}-z^n}{n'-n}-S(n,\alpha)z^n\ln z}{N(n,\alpha)}\right] \tag{42b}$$

as $2\alpha\to2\alpha^*$, n and $n'\to n^*$, $[(S(n',\alpha)-S(n,\alpha))/(n'-n)]\to W(n^*,\alpha^*)$, therefore all the fractions in (42) become the indeterminate mode of 0/0.

When (n,α) approaches (n^*,α^*) (where $n^*>0$) along the curve of $Q(\lambda,\alpha)=0$ ($\lambda\neq0$), it can be shown that

$$\lim_{\substack{2\alpha\to2\alpha^*\\(n,n'\to n^*)}}\frac{dn'}{d\alpha}\bigg/\frac{dn}{d\alpha}=-1 \tag{43a}$$

When (n,α) approaches $(0,\alpha_*)$ along the curve of $Q(\lambda,\alpha)=0$ ($\lambda\neq0$), n' equals zero and it can be shown that

$$\lim_{\substack{2\alpha\to2\alpha_*\\(n\to0)}}\frac{dn}{d\alpha}=-\frac{2}{\alpha_*} \tag{43b}$$

When (n,α) approaches $(0,\alpha_*)$ along the straight line $\lambda=0$, n equals zero and it can be shown that

$$\lim_{\substack{2\alpha\to2\alpha_*\\(n'\to0)}}\frac{dn'}{d\alpha}=-\frac{2}{\alpha_*} \tag{43c}$$

By applying L'Hospital's rule to take the limit $2\alpha=2\alpha^*$ in eqn (42) and making use of eqn (43), it can be proved that the limitation of (42) is the secondary paradox solution (22) (by replacing n, α with n^*, α^* and taking $A=E=0$). Hence (42) is bounded when $Q(n,\alpha)=0$ and $N(n,\alpha)$ very close to zero, which can be called the modified paradox solution.

For the special case $n = 0$ and 2α very close to $2\alpha_*$, if we take $n = 0$ and choose $L = -i[(p+2\alpha q)\cos(n'+2)\alpha/N(0,\alpha)]$ in (41), then superimpose it to the initial paradox solution (19) (by taking $n = 0$ and $A = 0$), we can obtain the result of Ting (1984), but he takes it as the modified classical solution.

According to the discussion above, when $Q(n,\alpha) = 0$, the bounded paradox solutions of antisymmetric deformations include:

(1) the initial paradox solution (19), which is suitable for the case $N(n,\alpha) \neq 0$ and not close to zero;
(2) the secondary paradox solution (22), which is suitable for the case $N(n,\alpha) = 0$;
(3) the modified paradox solution (42), which is suitable for the case $N(n,\alpha)$ very close to zero.

7.2. *The modified classical solutions* (*when* Q(n, α) *very close to zero*)

For the classical solution (18), when $Q(n,\alpha)$ very close to zero, the point (n,α) is certainly in the neighbouring region of the curve of $Q(\lambda,\alpha) = 0$ ($\lambda \neq 0$) and of the straight line $\lambda = 0$. Similar to symmetric deformations, there are also two possible situations: one is that there exists a wedge angle $2\alpha_0$ which is near 2α and satisfies $Q(n,\alpha_0) = 0$, the other is that there exists a constant $n_0(\geqslant 0)$ which is near n and satisfies $Q(n_0,\alpha) = 0$, we mainly study the first situation, and it can be observed easily from Fig. 2 that the point (n,α) which only belongs to the second situation corresponds to: (1) 2α equal or very close to 2π, n very close to but smaller than the semi-integer or n very close to but larger than the integer. (2) n very close to zero, 2α not close to $2\alpha_*$.

7.2.1. *When* N(n, α₀) ≠ 0 *and not close to zero*. In this case (n,α_0) is not near (n^*,α^*), and it can be seen from Fig. 2 that on the curve of $Q(\lambda,\alpha) = 0$ there always exists a point (λ,α) near (n,α) as 2α approaches $2\alpha_0$ while n is fixed, hence the homogeneous solution (29b) exists, in which we choose $L = -i[Y(p,q,\lambda,\alpha)/Q(n,\alpha)]$ and superimpose it to the classical solution (18), the result is

$$\Phi(z) = i\frac{Y(p,q,n,\alpha)z^n - Y(p,q,\lambda,\alpha)z^\lambda}{Q(n,\alpha)}$$

$$\Psi(z) = -i\left[X(p,q,n,\alpha)z^n + \frac{Y(p,q,n,\alpha)S(n,\alpha)z^n - Y(p,q,\lambda,\alpha)S(\lambda,\alpha)z^\lambda}{Q(n,\alpha)}\right] \quad (44)$$

as $2\alpha \to 2\alpha_0$, $\lambda \to n$, and it can be demonstrated that

$$\lim_{\substack{2\alpha\to 2\alpha_0 \\ (\lambda\to n)}} \frac{d\lambda}{d\alpha}\Bigg/\frac{dQ(n,\alpha)}{d\alpha} = -\frac{1}{N(n,\alpha_0)} \quad (45)$$

By applying L'Hospital's rule and eqn (45), it is shown that the limitation of (44) is the initial paradox solution (19) (by replacing α with α_0 and taking $A = -i[\alpha_0 X(p,q,n,\alpha)/N(n,\alpha_0)]$), thus the modified classical solution (44) is bounded.

For the special case $2\alpha_0 = \pi$, $n = 1,2,3,\ldots$ and $2\alpha_0 = 2\pi$, $n = 1/2, 1, 3/2, 2, \ldots$, (44) is coincident with the result of Wang (1986).

7.2.2. *When* n = n* > 0, 2α₀ = 2α*. In this case $N(n,\alpha_0) = 0$, similar to symmetric deformations, we construct the modified solutions under the following two different circumstances:

(a) As 2α approaches $2\alpha^*$ in the direction of turning towards π or 2π, from Fig. 2 it is observed that there exists a pair of complex numbers η and $\bar{\eta}$ which are near n^* and satisfy $Q(\eta,\alpha) = 0$ and $Q(\bar{\eta},\alpha) = 0$, hence we can construct the homogeneous solution from (29a) as below

$$\Phi(z) = L_1\frac{z^\eta+z^{\bar\eta}}{2}+L_2\frac{z^\eta-z^{\bar\eta}}{\eta-\bar\eta}$$

$$\Psi(z) = -L_1\frac{S(\eta,\alpha)z^\eta+S(\bar\eta,\alpha)z^{\bar\eta}}{2}-L_2\frac{S(\eta,\alpha)z^\eta-S(\bar\eta,\alpha)z^{\bar\eta}}{\eta-\bar\eta} \tag{46}$$

where L_1 and L_2 are arbitrary imaginary constants.
We choose

$$L_1 = -i\frac{Y(p,q,n^*,\alpha)}{Q(n^*,\alpha)} \quad\text{and}\quad L_2 = i\frac{Y(p,q,n^*,\alpha)\left(\dfrac{\eta+\bar\eta}{2}-n^*\right)}{Q(n^*,\alpha)}$$

in (46) and superimpose it to the classical solution (18), the result is

$$\Phi(z) = \frac{1}{2}iY(p,q,n^*,\alpha)\frac{(z^{n^*}-z^\eta)+(\eta-n^*)z^\eta\dfrac{z^{\bar\eta-\eta}-1}{\bar\eta-\eta}}{Q(n^*,\alpha)}$$

$$+\frac{1}{2}iY(p,q,n^*,\alpha)\frac{(z^{n^*}-z^{\bar\eta})+(\bar\eta-n^*)z^{\bar\eta}\dfrac{z^{\eta-\bar\eta}-1}{\eta-\bar\eta}}{Q(n^*,\alpha)} \tag{47a}$$

$$\Psi(z) = -iX(p,q,n^*,\alpha)z^{n^*}-\frac{1}{2}iY(p,q,n^*,\alpha)$$

$$\cdot\frac{[S(n^*,\alpha)z^{n^*}-S(\eta,\alpha)z^\eta]+(\eta-n^*)z^\eta\left[S(\bar\eta,\alpha)\dfrac{z^{\bar\eta-\eta}-1}{\bar\eta-\eta}+\dfrac{S(\bar\eta,\alpha)-S(\eta,\alpha)}{\bar\eta-\eta}\right]}{Q(n^*,\alpha)}$$

$$-\frac{1}{2}iY(p,q,n^*,\alpha)$$

$$\cdot\frac{[S(n^*,\alpha)z^{n^*}-S(\bar\eta,\alpha)z^{\bar\eta}]+(\bar\eta-n^*)z^{\bar\eta}\left[S(\eta,\alpha)\dfrac{z^{\eta-\bar\eta}-1}{\eta-\bar\eta}+\dfrac{S(\eta,\alpha)-S(\bar\eta,\alpha)}{\eta-\bar\eta}\right]}{Q(n^*,\alpha)} \tag{47b}$$

(b) As 2α approaches $2\alpha^*$ in the direction of deviating from π or 2π, from Fig. 2 it is observed that there exists two real numbers λ and λ' which are near n^* and satisfy $Q(\lambda,\alpha) = 0$ and $Q(\lambda',\alpha) = 0$, hence we can construct the homogeneous solution from (29b) as below

$$\Phi(z) = L_1z^\lambda+L_2\frac{z^{\lambda'}-z^\lambda}{\lambda'-\lambda}$$

$$\Psi(z) = -L_1S(\lambda,\alpha)z^\lambda-L_2\frac{S(\lambda',\alpha)z^{\lambda'}-S(\lambda,\alpha)z^\lambda}{\lambda'-\lambda} \tag{48}$$

where L_1 and L_2 are arbitrary imaginary constants.
We choose

$$L_1 = -i\frac{Y(p,q,n^*,\alpha)}{Q(n^*,\alpha)} \quad\text{and}\quad L_2 = i\frac{Y(p,q,n^*,\alpha)(\lambda-n^*)}{Q(n^*,\alpha)}$$

in (48) and superimpose it to the classical solution (18), the result is

$$\Phi(z) = iY(p,q,n^*,\alpha)\frac{(z^{n^*}-z^{\lambda})+(\lambda-n^*)z^{\lambda}\dfrac{z^{\lambda'-\lambda}-1}{\lambda'-\lambda}}{Q(n^*,\alpha)}$$

$$\Psi(z) = -iX(p,q,n^*,\alpha)z^{n^*}-iY(p,q,n^*,\alpha)$$

$$\cdot\frac{[S(n^*,\alpha)z^{n^*}-S(\lambda,\alpha)z^{\lambda}]+(\lambda-n^*)z^{\lambda}\left[S(\lambda',\alpha)\dfrac{z^{\lambda'-\lambda}-1}{\lambda'-\lambda}+\dfrac{S(\lambda',\alpha)-S(\lambda,\alpha)}{\lambda'-\lambda}\right]}{Q(n^*,\alpha)} \tag{49}$$

as $2\alpha\to2\alpha^*$, for (47) η and $\bar\eta\to n^*$, and it can be shown that $(\mathrm{d}\bar\eta/\mathrm{d}\alpha)/(\mathrm{d}\eta/\mathrm{d}\alpha)\to-1$; for (49) λ and $\lambda'\to n^*$, and it can be shown from (43a) that $(\mathrm{d}\lambda'/\mathrm{d}\alpha)/(\mathrm{d}\lambda/\mathrm{d}\alpha)\to-1$. By applying L'Hospital's rule twice in succession, it is shown that the limitations of (47) and (49) are both the secondary paradox solution (22) (by replacing n, α with n^*, α^* and taking $A=E=0$), hence the modified classical solutions (47) and (49) are bounded.

7.2.3. *When* $\mathrm{N}(n,\alpha_0)$ *very close to zero*. In this case (n,α_0) is near (n^*,α^*) (including $(0,\alpha_*)$), and it can be observed from Fig. 2 that on the curve of $Q(\lambda,\alpha)=0$ there exists another point (n',α_0) which is also near (n^*,α^*). Here we make a discussion for $n^*>0$ and $n^*=0$, respectively.

7.2.3.1. (n,α_0) *near* (n^*,α^*) *where* $n^*>0$.

(a) As 2α approaches $2\alpha_0$ in the direction of turning towards π or 2π, the modified classical solutions can be constructed under the following three different circumstances:

(1) If 2α is not so close to $2\alpha_0$ as $2\alpha^*$, there exists a pair of complex numbers η and $\bar\eta$ which are near n^* and satisfy $Q(\eta,\alpha)=0$ and $Q(\bar\eta,\alpha)=0$, adopting the same method as in 7.2.2(a), we obtain

$$\Phi(z) = iY(p,q,n,\alpha)\frac{\left(z^n-\dfrac{z^{\eta}+z^{\bar\eta}}{2}\right)+\left(\dfrac{\eta+\bar\eta}{2}-n\right)\dfrac{z^{\bar\eta}-z^{\eta}}{\bar\eta-\eta}}{Q(n,\alpha)}$$

$$\Psi(z) = -iX(p,q,n,\alpha)z^n-iY(p,q,n,\alpha)\frac{\left[S(n,\alpha)z^n-\dfrac{S(\eta,\alpha)z^{\eta}+S(\bar\eta,\alpha)z^{\bar\eta}}{2}\right]}{Q(n,\alpha)}$$

$$-iY(p,q,n,\alpha)\frac{\left(\dfrac{\eta+\bar\eta}{2}-n\right)\left[S(\bar\eta,\alpha)\dfrac{z^{\bar\eta}-z^{\eta}}{\bar\eta-\eta}+\dfrac{S(\bar\eta,\alpha)-S(\eta,\alpha)}{\bar\eta-\eta}z^{\eta}\right]}{Q(n,\alpha)} \tag{50a}$$

(2) If $2\alpha=2\alpha^*$, we take the limitation of eqn (50a) as $2\alpha\to2\alpha^*$ (thus, η and $\bar\eta\to n^*$), i.e.,

$$\Phi(z) = iY(p,q,n,\alpha^*)\frac{(z^n-z^{n^*})+(n^*-n)z^{n^*}\ln z}{Q(n,\alpha^*)}$$

$$\Psi(z) = -iX(p,q,n,\alpha^*)z^n-iY(p,q,n,\alpha^*)$$

$$\cdot\frac{[S(n,\alpha^*)z^n-S(n^*,\alpha^*)z^{n^*}]+(n^*-n)[S(n^*,\alpha^*)z^{n^*}\ln z+W(n^*,\alpha^*)z^{n^*}]}{Q(n,\alpha^*)} \tag{50b}$$

(3) If 2α is closer to $2\alpha_0$ than $2\alpha^*$, there exists two real numbers λ and λ' which are near n and n', respectively, and satisfy $Q(\lambda,\alpha)=0$ and $Q(\lambda',\alpha)=0$, adopting the same method as in 7.2.2(b), we obtain

$$\Phi(z) = iY(p,q,n,\alpha)\frac{(z^n - z^\lambda) + (\lambda - n)\dfrac{z^{\lambda'} - z^\lambda}{\lambda' - \lambda}}{Q(n,\alpha)}$$

$$\Psi(z) = -iX(p,q,n,\alpha)z^n - iY(p,q,n,\alpha)$$

$$\cdot\frac{[S(n,\alpha)z^n - S(\lambda,\alpha)z^\lambda] + (\lambda - n)\left[S(\lambda',\alpha)\dfrac{z^{\lambda'} - z^\lambda}{\lambda' - \lambda} + \dfrac{S(\lambda',\alpha) - S(\lambda,\alpha)}{\lambda' - \lambda}z^\lambda\right]}{Q(n,\alpha)} \quad (50c)$$

it is noticed that the limitation of (50c) as $2\alpha \to 2\alpha^*$ (thus, λ and $\lambda' \to n^*$) is also (50b), therefore the modified solutions (50a–c) connect continuously at $2\alpha = 2\alpha^*$.

(b) As 2α approaches $2\alpha_0$ in the direction of deviating from π or 2π, we can construct the modified solution (50c).

As $2\alpha \to 2\alpha_0$, $\lambda \to n$ and $\lambda' \to n'$, by using L'Hospital's rule and eqn (45), it is shown that the limitation of (50c) is the modified paradox solution (42) (by replacing α with α_0), hence the modified classical solution (50) is bounded.

7.2.3.2. (n, α_0) *near* $(0, \alpha_*)$. It can be observed from Fig. 2 that $n' = 0$, and as 2α approaches $2\alpha_0$ while n is fixed, there always exist two real number λ (near n) and $\lambda'(=0)$ satisfying $Q(\lambda,\alpha) = 0$ and $Q(\lambda',\alpha) = 0$, hence the bounded modified solution can be obtained from (50c) by taking $\lambda' = 0$ if $2\alpha \neq 2\alpha_*$ (thus, $\lambda \neq \lambda'$) or from (50b) by taking $n^* = 0$, $\alpha^* = \alpha_*$ if $2\alpha = 2\alpha_*$ (thus, $\lambda = \lambda'$).

7.2.4. *When* 2α *equal or very close to* 2π, n *very close to but smaller than the semi-integer on* n *very close to but larger than the integer.* In this case it can be observed from Fig. 2 that on the curve of $Q(\lambda,\alpha) = 0$ there always exists a point (n_0, α) $(n_0 > 0)$ near (n, α), moreover, (n_0, α) is not near (n^*, α^*), thus, $N(n_0, \alpha) \neq 0$ and not close to zero. We take $\lambda = n_0$ and choose $L = -i[Y(p,q,n,\alpha)/Q(n,\alpha)]$ in the homogeneous solution (29b), then superimpose it to (18), the result is

$$\Phi(z) = iY(p,q,n,\alpha)\frac{z^n - z^{n_0}}{Q(n,\alpha)}$$

$$\Psi(z) = -i\left[X(p,q,n,\alpha)z^n + Y(p,q,n,\alpha)\frac{S(n,\alpha)z^n - S(n_0,\alpha)z^{n_0}}{Q(n,\alpha)}\right] \quad (51a)$$

as $n \to n_0$, the limitation of (51a) is the initial paradox solution (19) (by replacing n with n_0 and taking $A = 0$), hence the modified classical solution (51a) is bounded.

7.2.5. *When* n *very close to zero,* 2α *not close to* $2\alpha_*$. In this case there exists $n_0 = 0$ satisfying $Q(n_0, \alpha) = 0$, so the bounded modified classical solution can be obtained from (51a) by taking $n_0 = 0$, i.e.,

$$\Phi(z) = iY(p,q,n,\alpha)\frac{z^n - 1}{Q(n,\alpha)}$$

$$\Psi(z) = -i\left[X(p,q,n,\alpha) + Y(p,q,n,\alpha)\frac{S(n,\alpha)}{Q(n,\alpha)}\right]z^n \quad (51b)$$

In comparison with the classical solution (18), it is evident that the modified classical solution (51b) only contains an imaginary number more in $\Phi(z)$, thus, for the classical solution (18), only its displacements must be modified whereas its stresses need not be.

Now, for antisymmetric deformations, the modified particular solutions which remain bounded as the denominators of them approach zero have been constructed completely.

8. CONCLUSION

This paper completely solves the paradox problem of the wedge subjected to tractions proportional to $r^n (n \geqslant 0)$, the main results are as follows:

(1) We obtain the initial paradox solution (12) and (19), discover that the secondary paradox exists for them, and we also obtain the corresponding bounded solutions, namely, the secondary paradox solution (15) and (22).

(2) By superimposing the proper homogeneous solutions to the initial paradox solutions, we successfully construct the modified paradox solution (31) and (42), which are still bounded as the denominators of them approach zero. Hence the bounded paradox solutions include the initial paradox solutions, the secondary paradox solutions and the modified paradox solutions.

(3) By superimposing the proper homogeneous solutions to the classical solutions, we successfully construct the modified classical solutions in various cases, which are still bounded as the denominators of them approach zero. For symmetric deformations, they are the solution (33), (36) and (38)–(40); for antisymmetric deformations, they are the solution (44), (47) and (49)–(51).

(4) For the special case $n = 0$, 2α near and equal to π, 2π or $2\alpha_*$, the results of Dempsey (1981) and of Ting (1984) can be derived from the solutions obtained in the present paper, so can the results of Wang (1986) for the special case 2α near and equal to π, $n = 1, 2, 3, \ldots$ and 2α near and equal to 2π, $n = 1/2, 1, 3/2, 2, \ldots$.

Acknowledgement—This work is supported by the Natural Science Foundation of Jiangxi Province and the Youth Foundation of Nanchang University.

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